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ON THE ADDITION OF MOTIONS RELATIVE TO DEFORMABLE REFERENCE SYSTEMS

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L.I. SEDOV
(Moscow)

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We consider within the framework of Newtonian mechanics the kinematic problem of motion of some individual point M relative to two arbitrary deformable reference systems with related individual points, using Lagrange coordinates x^α and ξ^α.

We denote by $N_1(x^\alpha)$ and $N_2(\xi^\alpha)$ the individual points in those two systems; these points at some arbitrarily recorded specified instant of time t coincide with point M. At the next instant $t + dt$ points N_1, N_2 and M separate (see Fig. 1). It is possible to determine the infinitely small displacement vectors $\mathbf{N_1M} = d\mathbf{r}$, $\mathbf{N_1N_2} = d\mathbf{r}_0$, and $\mathbf{N_2M} = d\mathbf{r}_1$ of these, which form an infinitely small triangle.

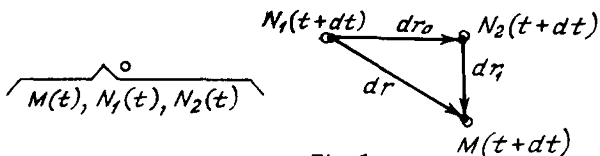

Fig. 1

These vectors and the triangle may be considered at instant $t + dt$ in the coordinate systems x^α, ξ^α, or any other system.

We introduce the coordinate bases $\text{Э}_\alpha = \partial\mathbf{r} / \partial x^\alpha$ in the system x^α and $\text{Э}_{\hat\alpha} = \partial\mathbf{r} / \partial\xi^\alpha$ in the system ξ^α.

The relative displacement vectors can be defined as follows:

$$d\mathbf{r} = dx^\alpha \text{Э}_\alpha = dx^{\hat\alpha} \text{Э}_{\hat\alpha}, \quad d\mathbf{r}_1 = d\xi^\alpha \text{Э}_{\hat\alpha}$$

$$d\mathbf{r}_0 = dx_0{}^\alpha \text{Э}_\alpha = dx_0^{\hat\alpha} \text{Э}_{\hat\alpha}$$

$$dr = dr_1 + dr_0 \tag{1}$$

In Newtonian mechanics time t at all individual points is absolute, identical, and may be assumed synchronised. We shall consider the reference system x^α, t to be that of the observer, and the reference system ξ^α, t to be that of the transport motion. We define the "absolute" velocity $\mathbf{v}_a$, the relative velocity $\mathbf{v}_r$ and the transport velocity $\mathbf{v}_t$ by formulas

$$\mathbf{v}_a = \frac{d\mathbf{r}}{dt} = \frac{dx^\alpha}{dt} \text{Э}_\alpha = \frac{dx^{\hat\alpha}}{dt} \text{Э}_{\hat\alpha} = v_a^{\hat\alpha} \text{Э}_{\hat\alpha}$$

$$\mathbf{v}_r = \frac{d\mathbf{r}_1}{dt} = \frac{d\xi^\alpha}{dt} \text{Э}_{\hat\alpha} = v_r^{\hat\alpha} \text{Э}_{\hat\alpha}$$

$$\mathbf{v}_t = \frac{d\mathbf{r}_0}{dt} = \frac{dx_0{}^\alpha}{dt} \text{Э}_\alpha = \frac{dx_0^{\hat\alpha}}{dt} \text{Э}_{\hat\alpha} = v_t^{\hat\alpha} \text{Э}_{\hat\alpha}$$

Using formula (1) and the universality of time at any instant t, we may write

$$\mathbf{v}_a(t) = \mathbf{v}_r(t) + \mathbf{v}_t(t) \tag{2}$$

Equality (2) represents the fundamental relation between various velocities of the moving point M relative to any deformable reference systems.

Let us now consider the accelerations determined by derivatives of velocity vectors with respect to time of the moving point M. On the assumption that the vectors of basis Э_α are time invariant, the absolute acceleration of point M in the reference system x^α, t is determined by the formula

$$(\mathbf{a}_a)_M = \left(\frac{d\mathbf{v}_a}{dt}\right)_M = \frac{dv_a^{\ \alpha}}{dt}\,\text{Э}_\alpha = \left(\frac{\partial v_a^{\ \alpha}}{\partial t} + v_a^{\ \beta}\nabla_\beta v_a^\alpha\right)\text{Э}_\alpha \tag{3}$$

and the relative acceleration of point M in the reference system ξ^α by

$$(\mathbf{a}_r)_M = \left(\frac{d\mathbf{v}_r}{dt}\right)_M = \frac{\partial \mathbf{v}_r}{\partial t} + \frac{\partial \mathbf{v}_r}{\partial \xi^\beta}\frac{d\xi^\beta}{dt} = \left(\frac{\partial v_r^{\ \hat\alpha}}{\partial t} + v_r^{\ \hat\beta}\nabla_{\hat\beta} v_r^{\ \hat\alpha}\right)\hat{\text{Э}}_\alpha + \tag{4}$$

$$v_r^{\ \hat\alpha}\left(\frac{\partial \hat{\text{Э}}_\alpha}{\partial t}\right)_{\xi^\alpha} = (\mathbf{a}_r)_{N_2=M} + v_r^{\ \hat\alpha}\frac{\partial \mathbf{v}_t}{\partial \xi^\alpha} = (\mathbf{a}_r)_{N_2=M} + v_r^{\ \hat\alpha}\nabla_{\hat\alpha}\, v_t^{\ \hat\beta}\hat{\text{Э}}_\beta$$

Variation of the basis vectors relative to the observer's system is taken into account in the transport system, hence formulas

$$\left(\frac{\partial \hat{\text{Э}}_\alpha}{\partial t}\right)_{\xi^\alpha} = \frac{\partial}{\partial \xi^\alpha}\left(\frac{\partial \mathbf{r}_0}{\partial t}\right)_{\xi^\alpha} = \frac{\partial \mathbf{v}_t}{\partial \xi^\alpha} = \nabla_{\hat\alpha} v_t^{\ \hat\beta}\hat{\text{Э}}_\beta$$

are valid. The relative acceleration $(\mathbf{a}_r)_{N_2}$ is determined in basis $\hat{\text{Э}}_\alpha$ as well as the absolute acceleration in Э_α.

We define the transport acceleration for point M moving relative to the reference system ξ^α, t, by formula

$$(\mathbf{a}_t)_M = \left(\frac{d\mathbf{v}_t}{dt}\right)_M = \left(\frac{d\mathbf{v}_t}{dt}\right)_{N_2=M} + \frac{\partial \mathbf{v}_t}{\partial \xi^\alpha}\frac{d\xi^\alpha}{dt} = (\mathbf{a}_t)_{N_2=M} + v_r^{\ \hat\alpha}\nabla_{\hat\alpha} v_t^{\ \hat\beta}\hat{\text{Э}}_\beta \tag{5}$$

Taking into account equalities (3) — (5) we differentiate (2) and obtain

$$(\mathbf{a}_a)_M = (\mathbf{a}_r)_M + (\mathbf{a}_t)_M = (\mathbf{a}_r)_{N_2=M} + (\mathbf{a}_t)_{N_2=M} + 2v_r^{\ \hat\alpha}\nabla_{\hat\alpha} v_t^{\ \hat\beta}\hat{\text{Э}}_\beta \tag{6}$$

The vector of the last term may be expressed in any basis, in particular in basis Э_β, taking also into consideration that

$$\nabla_{\hat\alpha} v_t^{\ \hat\beta}\hat{\text{Э}}_\beta = \nabla_{\hat\alpha} v_{\hat\beta t}\hat{\text{Э}}^{\beta} = \nabla_\alpha v_{\beta t}\text{Э}^\beta = (e_{\alpha\beta} + \omega_{\alpha\beta})\,\text{Э}^\beta$$

$$e_{\alpha\beta} = {}^1\!/_2(\nabla_\alpha v_{\beta t} + \nabla_\beta v_{\alpha t}), \quad \omega_{\alpha\beta} = {}^1\!/_2(\nabla_\alpha v_{\beta t} - \nabla_\beta v_{\alpha t})$$

Formula (6) which determines the acceleration $(\mathbf{a}_a)_M$ may be written in the following final form

$$(\mathbf{a}_a)_M = (\mathbf{a}_r)_M + (\mathbf{a}_t)_M = (\mathbf{a}_r)_{N_2=M} + (\mathbf{a}_t)_{N_2=M} + 2v_r^{\ \alpha}(e_{\alpha\beta} + \omega_{\alpha\beta})\,\text{Э}^\beta \tag{7}$$

Formula (7) is a generalization of the " Coriolis theorem". In it $\omega_{\alpha\beta}$ is an antisymmetric tensor that represents the instantaneous angular velocity of deformation axes at point N_2 for the motion of the transport reference basis Э_α and $e_{\alpha\beta}$ is the tensor of deformation rates of reference $\hat{\text{Э}}_\alpha$ in the motion of the transport system relative to the basis reference Э_α in the observer's system.

The Coriolis theorem is usually established for absolutely rigid reference systems with bases $\ni_\alpha$ and $\ni_{\hat{\alpha}}$. Since in this case $e_{\alpha\beta} \equiv 0$, formula (7) is of the conventional form.

In practice, when dealing with dynamic systems, the observer's system is usually understood as an inertial coordinate system, hence the use of the term " absolute acceleration".

Note that in the general case all quantities in formula (7) are exactly determined and it is not implied that the observer's system is undeformable, although the definition of absolute acceleration by formula (3) is based on the assumption of time invariance of vectors of basis $\ni_\alpha$ (these vectors can arbitrarily vary with respect to coordinates).

Compared with the usual proof of the Coriolis theorem the above derivation using the simplest concepts of tensor analysis and almost without any calculations yields a more general result in a more general situation with an intuitive demonstration of the origin and nature of the "additional" acceleration in formula (7).

Separation of the absolute and relative motions of the moving point M is related to the introduction of the transport motion reference system which may be applied generally and in a holonomic manner, as well as locally for each position of the moving point M and, generally in a nonholonomic manner.

The intermediate reference system of transport motion may be considered as a system of individual points with frozen in coordinate lines. Such system of points can form a material medium, such as solid body, liquid, gas, plasma, cloud, dust, or some idealized object based on some special mathematical constructions.

For instance, in the case of a system of finite masses or material points which obey Newton's law of attraction it is possible to consider a reference system consisting of the trajectories of sample points that fill a certain volume. The corresponding Lagrangian coordinate system for the transport motion may be constructed as follows. The equations of motion of sample particles or material points in a gravitational field in an inertial reference system x^α are of the form

$$\frac{d^2\mathbf{r}}{dt^2} - \mathbf{g} = \mathbf{f} \tag{8}$$

where $\mathbf{g}\,(x^2, x^2, x^3, t) = \operatorname{grad} U$ is the acceleration of gravity, U is the specific potential of gravitation forces, and $\mathbf{f}$ is an external force per unit of mass. Force $\mathbf{f}$ may be nonzero owing to the presence of electromagnetic forces and the nongravitational forces of interaction between a given particle and neighboring and other particles. When the motion of free particles is subjected only to gravitational attraction between each other, $\mathbf{f} = 0$.

Equations (8) are valid for dust particles in a gravitational field or for sensitive elements, such as accelerometer "small balls" placed at some points of a body and supported by springs or some other means which make possible the measurement of vector $\mathbf{f}$.

A deformable reference system for the transport motion in Lagrangian coordinates ξ^α can be obtained by solving the Cauchy problem for Eq. (8) for specified initial velocities $\mathbf{v}_0 = \xi^\alpha \ni_\alpha$ at $t = t_0$ at points of a particular volume.

It will be readily seen that, if $\mathbf{v}_0 = \operatorname{grad} \varphi$ is assumed, then the potential character of the hydrodynamic velocity fiels for the motion of sample points that form the transport reference system relative to the inertial reference system x^α, where $\mathbf{r} = x^\alpha \mathbf{\vartheta}_\alpha$, remains unchanged for $\mathbf{f} = 0$ and all $t \neq t_0$.

It is, thus, possible to devise in celestial mechanics an accompanying system of transport motion in which at all points the equality $\omega_{\alpha\beta} = 0$ is valid, although in the general case the deformation rate tensor is nonzero, i.e. $e_{\alpha\beta} \neq 0$.

If the motion of the transport reference system is translational, the equalities $\omega_{\alpha\beta} = e_{\alpha\beta} = 0$ are valid.

In the case of flight of bodies in a gravitational field it is advisable to introduce locally reference systems for the transport motion.

On the basis of formulas (7) and (8) we obtain

$$\mathbf{a}_r + 2v_r^{\alpha}(e_{\alpha\beta} + \omega_{\alpha\beta})\,\mathbf{\vartheta}^{\beta} = \mathbf{f}$$

It is clear that generally $\mathbf{f} \neq \mathbf{a}_r$ but $\mathbf{f} = \mathbf{a}_r$ when $\mathbf{v}_r = 0$ or the locally introduced transport reference system relative to the inertial reference system is translational with acceleration $\mathbf{g}$ in a given point. Such analysis is also valid in cases in which acceleration $\mathbf{g}$ at neighboring points is different.

Translated by J. J. D.